Background risk and quantum calculus

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Abstract

Infinitesimal calculus is heavily used in decision making analysis. This paper demonstrates that the application of quantum calculus in analysing preferences choice directly introduces background risk and its effects on risk-aversion, subjective probabilities and moment preferences. Quantum calculus provides another approach to the mathematical treatment of decision making, namely analysis of utility preferences.

Keywords: q-Calculus, Utility, Background risk, Unfair lotteries

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Introduction

Infinitesimal calculus is heavily used in the mathematical treatment of decision making in economics and finance. In the analysis of utility preferences, infinitesimal differentiation underlies the assumption that the marginal utility associated with a given outcome is precise. In other words, no uncertainty (or ambiguity) is associated with it. However, the literature indicates that to every outcome there is some degree of ambiguity that is associated with it. Hence, in addition to risk aversion there is also an ambiguity (or uncertainty) aversion, because *ex-ante* probabilities are unknown (Gilboa & Schmeidler, 1989)[1].

Alternatively, this imprecision has been attributed to the presence of undesirable lotteries that constitute background risk (see: Samuelson (1963); Pratt and Zeckhauser (1987); Kimball (1993); Gollier and Pratt (1996)). In the presence of such risk, an investment is a combination of two lotteries: one that is fair (can be hedged completely) and another that is unfair (cannot be hedged completely). It is natural to link the latter with ambiguity, because the task of hedging away the risks associated with an investment becomes extremely challenging when the probabilities are *ex-ante* unknown.

The analysis of background risk and the ambiguity that relates to it is widely discussed in the academic literature. This paper provides a new approach to analyse background risk by applying quantum calculus (henceforth, q-calculus), which directly incorporates the notion of ambiguity in the analysis of utility preferences. At first, an analogy between the concept of background risk (Gollier, 2004) and (quantum) q-derivatives is made. The application of the latter introduces a quantum parameter that measures the degree of imprecision in assigning a utility to a given outcome due to background risk.

Incorporating q-calculus in the analysis of choice preference affects at first the subjective probability that is required by an agent to accept a given lottery. Basically, the latter turns out to be distorted (i.e.: does not sum to one) when the utility assignment is imprecise. This phenomenon is well known in physics and has been attributed to non-extensive systems. In these systems, composing elements bear long-range interactions and long-term memory (Borland, 2005). Decision theory has recognised that distorted probabilities are possible when there is ambiguity (uncertainty) associated with them. Izhakian (2012) discusses two kind of distorted probabilities, one that sum to a number higher then one and another that sums to a number lesser then one. The first kind corresponds to "ambiguity loving " and the second to an "ambiguity aversion" behaviour. Application of q-calculus in the simple case of only two possible outcomes leads naturally to a subjective probability whose sum is greater then one (with a different interpretation than in Itzhakian (2012)). In this paper, deformation is attributed to the imprecision caused by background risk. Nevertheless, the distorted probabilities are power normalised in a similar manner as in Tsallis (1988) and Wang (2002).

[1]Since probability functions provide a description of utility preferences (see: Berhold (1973); Abbas (2006)) it is possible to relate ambiguity aversion to the imprecise measure of marginal utility.

At last, moment preferences is analysed. Brockett and Kahane (1992) demonstrate that given a different structure of probabilities, risk is insufficient in determining preferences between two distinct lotteries. They also demonstrate that the three first statistical moments (mean, variance and skewness) are also not sufficient in summarising agents preferences. Their work is complemented using the q-calculus framework. As with subjective probabilities, the structure of the probabilities are altered when ambiguity is explicitly introduced.

The use of q-calculus framework is motivated by an analogy drawn from the physical relation of q-calculus and background risk in decision theory. In physics, q-calculus is related to Heisenberg (1927) uncertainty principal (Swamy, 2003). In summary, it states that a pair of physical properties of an element cannot be simultaneously known at the same time. In case of background risk (and the ambiguity related to it) we have a similar situation, because it is not tradable and hard to quantify. Therefore, it is possible to assume that the notion of "background risk" is closely associated to the incompleteness of the set of state preferences and therefore to probabilities being *ex-ante* unknown.

This paper is divided into four sections. First, we briefly present elements of q-calculus in the analysis of utility functions. The second section derives the subjective density by applying the q-Taylor expansion to a given utility function. We also examine, using a simple model the possible relation between state preference incompleteness and background (unfair) risk. The third section describes the effects of background risk on moment preferences. At last, we conclude.

1 Background risk and q-calculus

Background risk is stated in Gollier (2004) by the utility inequality:

$$u(x) \geq u(x+h) \tag{1}$$

Where: $u(\cdot)$ is a utility function, x is some identified outcome and h ($h \leq 0$) is the expected value of an unfair background risk. h is non-tradable and hard to quantify. In its presence, the task of utility assignment to a set preferences becomes imprecise[2]. To circumvent this difficulty, Kihlstrom et al. (1981); Gollier and Pratt (1996) and Gollier (2004) propose to change the measure of preferences:

$$v(x) = u(x+h) \ \ \textit{such that} \ \ v^n(x) = u^n(x+h) \tag{2}$$

The effect of background risk in equation (2) is similar to the effect of a change of a preferences measure, i.e. change of measure from $u(\cdot)$ to $v(\cdot)$. However, Gollier (2004) and others have shown that not all characteristics of $u(\cdot)$ are transferred into $v(\cdot)$ when the change of measure takes place. Alternatively, without a change of the utility measure, we have from equation (1):

$$u(x+h) - u(x) \leq 0 \tag{3}$$

[2]see also: Schaden (2002); Accardi and Boukas (2007); Melnyk and Tuluzov (2008)

Which implies that a background risk (h) (representing a loss of financial wealth, for example) negatively impacts agents utility. Dividing equation (3) by h, yields the quantum derivative operator. Also known as the h-derivative that converges to the infinitesimal derivative as $h \to 0$.

$$D_h\left[u(x)\right] = \frac{u(x+h) - u(x)}{h}, \quad h \leq 0 \tag{4}$$

Equation (4) is interpreted as a quantum version of the marginal utility or the q or h-marginal utility[3]. For example, assuming an exponential utility function $u(x) = 1 - e^{-\gamma x}$, the h-marginal utility according to equation (4) is:

$$\begin{aligned} D_h\left[u(x)\right] &= e^{-\gamma x}\frac{1+e^{-\gamma h}}{h} \\ D_{h\to 0}\left[u(x)\right] &= \gamma e^{-\gamma x} \end{aligned} \tag{5}$$

The application of the h-derivative does not affect the sign of n'th derivative. It indicates a decrease of agent welfare per unit of background risk (h). Therefore providing an interpretation of marginal utility when the loss (in absolute value) is incurred by a latent (or background) risk associated with an unfair lottery. In other words, the interpretation of the q-marginal utility is bi-dimensional. A first dimension treats the effect of a change in x on agents utility and the other treats the effect of background risk on an agent ability to precisely assign a utility to a given preference. Further, if the second quantum derivative decreases, it means that the effect of background risk on agents' precision in utility assignment is reduced as wealth increases. For example, the second derivative of $u(x) = 1 - e^{-\gamma x}$ is:

$$D_h^{(2)}\left[u(x)\right] = -\left(\frac{1+e^{-\gamma h}}{h}\right) e^{-\gamma x} \tag{6}$$

In q-calculus the h-derivative is equivalent to Jacksons q-derivative such that $q = e^h$ ($h \leq 0$) (Kac and Cheung, 2002). The Jackson q-derivative is presented in a heuristic manner. Given q and h, equation (3) is re-written such that:

$$u(qx) - u(x) \leq 0,\ q \in [0,1] \tag{7}$$

Where q is as a fraction defining the residual terminal wealth x when background risk is a accounted for (being latent, however). Dividing the above by $x(q-1)$ yields the q-derivative that is equivalent to the h-derivative in equation (4). Specifically:

$$D_q\left[u(x)\right] = \frac{u(qx) - u(x)}{x(q-1)} \tag{8}$$

[3]Henceforth we shall use the term q-adjective to describe a quantum version a quantity or operator.

The interpretation of D_q is similar to that of the D_h operator. Applying it to some utility functions (such as the power or logarithmic utility functions) makes the analysis simple. For example, considering a power utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, the application of the q-derivative yields:

$$\begin{aligned} D_q\left[u(x)\right] &= \frac{[1-\gamma]_q}{1-\gamma} x^{-\gamma} \\ [1-\gamma]_q &= \frac{q^{1-\gamma}-1}{q-1} \\ D_{q\to 1}\left[u(x)\right] &= x^{-\gamma} \end{aligned} \tag{9}$$

The term in brackets with index q, $[1-\gamma]_q$, is the q-analog of an integer or q-integer. Figure 1 plots equation (9) given $\gamma = 3$ and for different values of q.

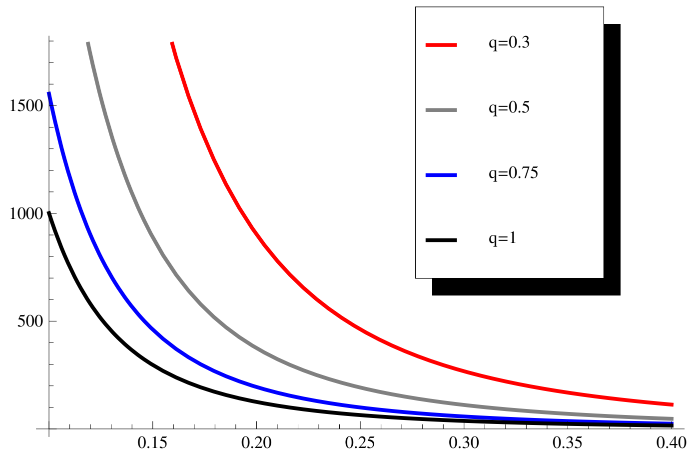

Figure 1: *q-marginal utility* of $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ at different levels of q

Equation (9) reveals that as q decreases, q-marginal utility increases for all levels of x. Moreover, it becomes more sensitive to changes in wealth, declining as wealth increases. Figure 1 raises a question as to why the q-marginal utility shifts upward when q decreases (in other words, as background risk becomes more significant). A possible explanation is that, given some level of q, an agent attaches more value to an increase in wealth. Furthermore, recalling the interpretation of the *q-marginal utility*, it also represents the decrease of agents welfare (in absolute value) due to background risk. Thus, the shifts in the q-marginal utility in figure 1 are representing this decrease in agents welfare.

In light of the above, the second q-derivative and the q-absolute risk aversion (using the Arrow-Pratt index of risk aversion) are related to each other. Applying again the q-derivative on a given utility function $u(\cdot)$, yields:

$$D_q^{(2)}\left[u(x)\right] = \frac{\frac{1}{1-q}u(q^2x) - \left(\frac{q}{q-1} + \frac{1}{x-1}\right)u(qx) + \frac{x}{x-1}u(x)}{qx^2(q-1)} \tag{10}$$

The second q-derivative of $u(\cdot)$ in equation (10) yields a nonlinear relationship with respect to q. Dividing the above equation with $D_q[u(x)]$ yields a q-absolute risk aversion index (denoted by $A_r^q(x)$):

$$A_r^q(x) = -\frac{D_q^{(2)}[u(x)]}{D_q[u(x)]} = \frac{\frac{1}{1-q}u(q^2x) - \left(\frac{q}{q-1} + \frac{1}{x-1}\right)u(qx) + \frac{x}{x-1}u(x)}{qx(u(x) - u(qx))} \tag{11}$$

For a power utility function we note that $A_r^q(x)$ explicitly depends on q. Applying the second derivative on $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ yields, respectively, the following results for $D_q^{(2)}[u(x)]$ and $A_r^q(x)$:

$$\begin{aligned} D_q^{(2)}[u(x)] &= \frac{[1-\gamma]_q}{1-\gamma} D_q[u(x)] = \frac{[1-\gamma]_q[-\gamma]_q}{1-\gamma} x^{-\gamma-1} \\ A_r^q(x) &= -\frac{D_q^{(2)}[u(x)]}{D_q[u(x)]} = -\frac{[-\gamma]_q}{x}, \quad \lim_{q\to 1} A_r^q(x) = \frac{\gamma}{x} \end{aligned} \tag{12}$$

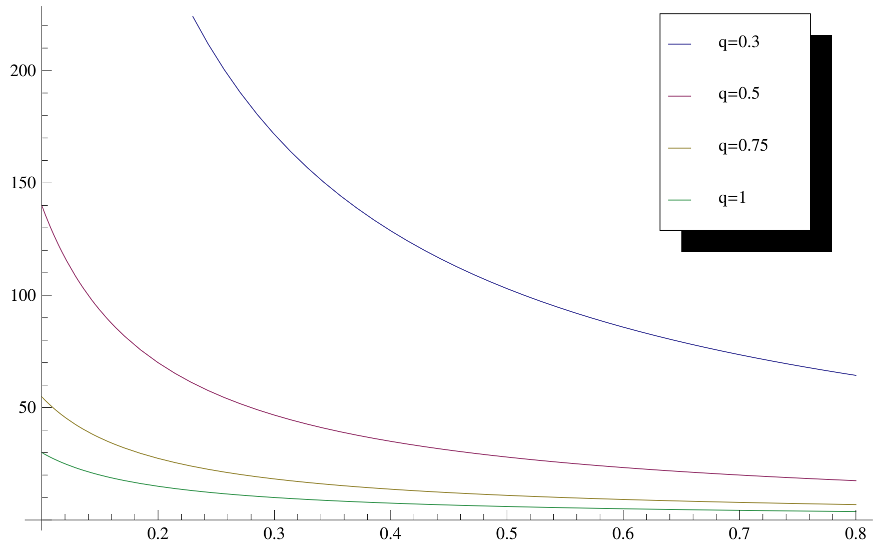

Figure 2: *q-absolute risk aversion* of $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ at different levels of q

As a result, application of the q-derivative does not affect the decreasing absolute risk aversion (DARA) property of the power utility function. Figure 2 plots equation (12) for several values of q and $\gamma = 3$.

Figure 2 indicates that, for a power utility function, the q-absolute risk aversion shifts upward as q decreases (in other words, as background risk becomes more significant). It confirms the results indicated in previous research (Kimball, 1993), where given background risk, aversion to risk increases. Furthermore, similar to the q-marginal utility, q-absolute risk aversion becomes more sensitive as q decreases. This observation is concurrent with our interpretation of the q-marginal utility.

An additional expansion provides a q-utility equivalence to prudence, defined as the negative ratio of the third to the second derivative of a utility function. For a DARA utility, prudence decreases with wealth. Kimball (1993) shows that for these types of utility functions, a decrease in absolute prudence (due to increase in wealth) is stronger than decreasing absolute risk aversion. In q-calculus this relationship for an isoelastic utility function is however:

$$P_r^q = -\frac{D_q^{(3)}\left[u(x)\right]}{D_q^{(2)}\left[u(x)\right]} = -\frac{[-(\gamma+1)]_q}{x} \tag{13}$$

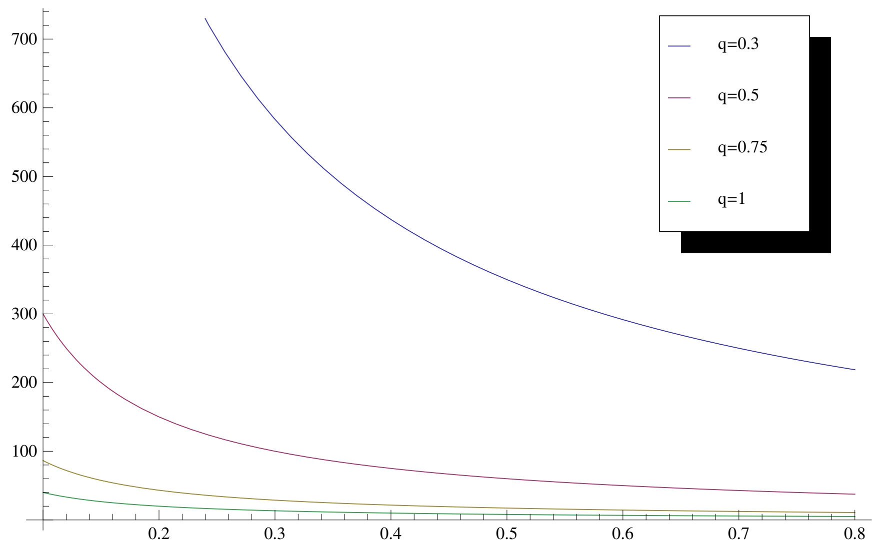

Figure 3: *q-absolute risk prudence* of $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ at different levels of q

Figure 3 (above) plots the q-absolute prudence for several values of q. Note that the same properties that were observed in the case of q-absolute risk aversion are observed in the q-absolute prudence. Furthermore, equation (13) indicates that a decreasing absolute prudence remains stronger than a decreasing absolute risk aversion as stipulated in Kimball (1993).

We have thus demonstrated how background risk is explicitly introduced when applying q-calculus analysis. It shows that as $q \to 0$ (or, $h \to -\infty$), the quantum parameters embodies background risk with risk-aversion increasing with change in outcomes. A similar conclusion is drawn for absolute prudence. The next section extends further the analysis to the subjective probability measure.

2 Incomplete subjective probability

2.1 The subjective probability

Incomplete (or distorted) probability density functions have gained much attention in physics (Wang, 2002; Yamano, 2004), information theory (Darooneh et al., 2010) and finance (Itzhakian, 2012). Performing a q-Taylor series approximation to a utility function highlights the deforming effects that background risk may have on the required subjective probability. In Kac and Cheung (2002) the q-Taylor approximation (or the q-analog of the Taylor approximation) is:

$$u(x) = \sum_{i=0}^{\infty} D_q^{(i)}\left[u(k)\right] \frac{(x-k)_q^i}{[i]_q!} \tag{14}$$

Where the functions, $(x-k)_q^i$ and $[i]_q!$, are (respectively) the q-polynomial and q-factorial functions, such that:

$$(x-k)_q^i = (x-k)(x-qk)...(x-q^{i-1}k), \quad i \geq 1 \tag{15}$$

$$[i]_q! = \frac{(q;q)_i}{(1-q)^i}, \quad (q;q)_i = \prod_{j=0}^{i}(1-q^j) \tag{16}$$

To illustrate how background risk distorts the subjective probability, consider a utility function and a possible wealth gain or loss y. The expected utility, in this case, is:

$$Eu(x) = pu(x+y) + (1-p)u(x-y) \tag{17}$$

Applying the q-Taylor approximation on $u(x+y)$ and $u(x-y)$ up to the second order, yields:

$$u(x+y) = u(x) + yD_q[u(x)] + q\frac{y^2}{[2]_q!}D_q^{(2)}[u(x)] \tag{18}$$

$$u(x-y) = u(x) - yD_q[u(x)] + q\frac{y^2}{[2]_q!}D_q^{(2)}[u(x)] \tag{19}$$

Inserting in the above the expectations given in equation (17) yields the subjective probability, p:

$$p = \frac{1}{2} - \frac{qy}{2[2]_q!}\frac{D_q^{(2)}[u(x)]}{D_q[u(x)]} \tag{20}$$

It is trivial to conjecture that the more risk averse an agent is, the higher will be his required subjective probability to accept this lottery. This is illustrated with two examples, one with the isoeslastic (generalised logarithmic) utility function and the other with the negative exponential utility.

Example 1 *Isoelastic (Generalized Logarithmic) Utility Function*

Consider the isoelastic utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ $(\gamma < 1)$ and $y = \phi x$ $(\phi \in [0,1])$, where ϕ is fraction of wealth at stake. In this case, equation (20) becomes:

$$\begin{aligned} p &= \frac{1}{2} - \frac{[-\gamma]_q}{2(1+q)}\phi q \\ \lim_{q\to 1} p &= \frac{1}{2} + \frac{\gamma}{4}\phi \end{aligned} \tag{21}$$

Figure 4 plots the probability surface that equation (21) yields, as a function of the background risk (q) and the risk aversion parameter (γ), for a predetermined level of percentage of wealth at stake (ϕ). It demonstrates that for any given level of risk aversion, the subjective probability is concave with respect to background risk, q. This concavity is further pronounced with an increasing risk aversion. Thus, the more risk averse an agent is, the more sensitive will he be to background risk.

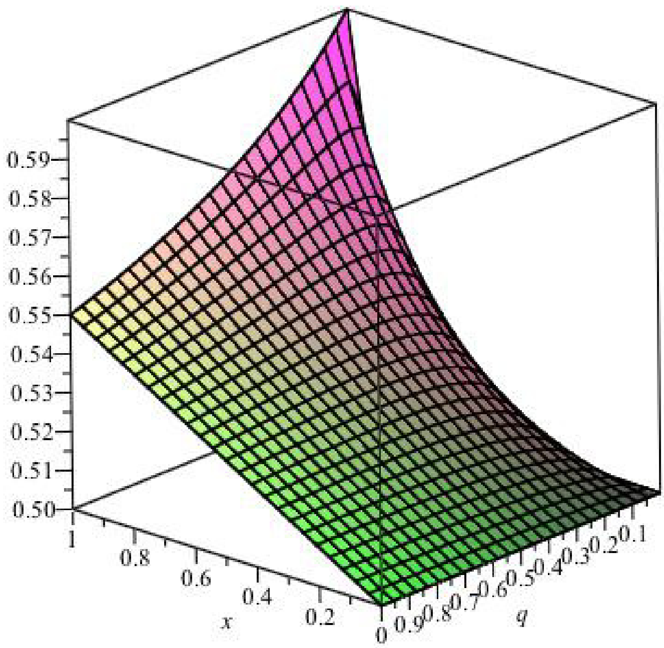

Figure 4: The probability surface of $p = \frac{1}{2} - \frac{[-\gamma]_q}{2(1+q)}\phi q$]

We further comment that for up to some threshold level of q (the inflection point where subjective probability begins to decrease with q), the subjective probability increases as the effects of background risk diminishes (or q increases). This specific phenomenon is more pronounced as risk aversion is decreasing. Thus, for extremely low levels of risk-aversion, the probability increases with q (in the case of an isoelastic utility). This suggests a possible substitution effect between risk-aversion and background risk. However, such a phenomenon is less important than risk aversion.

Example 2 *The Negative Exponential Function*

The h-derivative, defined in the first section, and the identity $h = \ln q$ are being applied for the derivation of the subjective probability. For an exponential utility function, the absolute risk aversion index (A_r^q) is:

$$A_r^q = \frac{q^{-\gamma} - 1}{\ln q}, \quad \lim_{q \to 1} \frac{q^{-\gamma} - 1}{\ln q} = \gamma \tag{22}$$

Then the subjective probability in equation (19) becomes:

$$\begin{gathered} p = \frac{1}{2} + \frac{\phi x}{2[2]_q!} \frac{q^{-\gamma} - 1}{\ln q} \\ \frac{\partial p}{\partial q} < 0, \quad \frac{\partial^2 p}{\partial q^2} < 0 \quad q \in [0, 1] \\ \lim_{q \to 1} p = \frac{1}{2} + \frac{x}{2}\phi x \end{gathered} \tag{23}$$

Thus, the subjective probability is also convex with respect to q. As background risk becomes less significant, an agents subjectively require less probability in order to enter in a lottery.

2.2 Deformed (Distorted) Subjective Probability

Both examples indicate that some combinations of initial wealth, background risk (q) and degree of risk aversion (γ), yield a subjective probability that is greater than one. This makes the set of identified probabilities non-additive. This is reasonable where there is ambiguity concerning the set of possible outcomes and their associated probabilities.

For instance, in example 1 γ was restricted to be less than one. However, if $\gamma = 3$ and the fraction of wealth at stake is $\phi = 20\%$, then it is sufficient for $q \leq 0.48$ for the subjective probability to be greater than one. While for $\gamma = 4$ and a same fraction of wealth at stake, it is sufficient for $q \leq 0.65$. In contrast, with ordinary calculus (given the same fractional initial wealth at stake) γ has to be higher than 10. This illustrates that under some conditions, the subjective probability might be deformed (i.e.: $p > 1$ in a binomial settings or summing to a quantity higher than one) even if risk-aversion is relatively low or at relatively normal levels. Figure 5 plots the subjective probability surface for $\gamma \in [1, 2]$ and different levels of q. It indicates that given the circumstances, an agent might require a subjective probability higher than one.

The problem of deformed probabilities is tightly linked to the problem information incompleteness. Explicitly, this incompleteness is present when there is an unavoidable and uninsurable background risk. This conforms LeRoy and Singell (1987) interpretation of Knight (1921) uncertainty, relating the latter with the presence of outcomes bearing adverse selection or moral hazard.

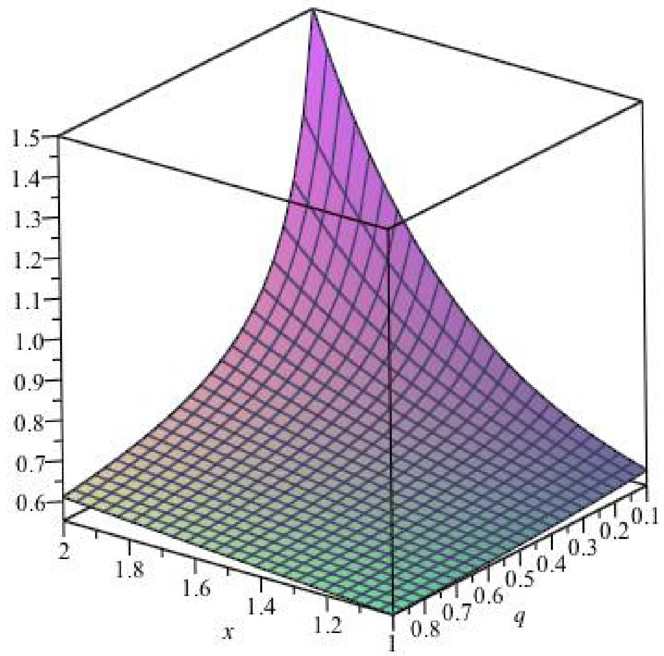

Figure 5: The probability surface of $p = \frac{1}{2} - \frac{[-\gamma]_q}{2(1+q)}\phi q \quad (\gamma \in [1, 2])$

To circumvent this problem, while including the possibility of a deformed distribution, we apply the same notion of incomplete statistics as in Wang (2002) where probabilities are "power" normalised so that they sum to one. This is consistent with theories generalising the expected utility theory (for example: Karmarkar (1979)). With a power transformation of these probabilities, equation (17) has the following form:

$$\tilde{E}u(x) = p_1^{\xi} u(x+y) + p_2^{\xi} u(x-y) \tag{24}$$

Where the tilde signifies that the expectation above is based on the deformed subjective probabilities. Furthermore, the probabilities p_1 and p_2 are such that:

$$p_1 + p_2 = \Omega \tag{25}$$

Thus, equation (24) becomes:

$$\tilde{E}u(x) = p_1^{\xi} u(x+y) + (\Omega - p_1)^{\xi} u(x-y) \tag{26}$$

Writing $\pi_1 = p_1^{\xi} = \frac{p_1}{\Omega}$ and $\pi_2 = (\Omega - p)^{\xi} = 1 - \frac{p_1}{\Omega}$, (the "pseudo-real effective probabilities") and using equation (18) and equation (26), yields the following equation:

$$\pi_1 \left(D_q [u(x)] + q\phi x \frac{D_q^{(2)} [u(x)]}{[2]_q!} \right) - \pi_2 \left(D_q [u(x)] - q\phi x \frac{D_q^{(2)} [u(x)]}{[2]_q!} \right) \quad (27)$$

With $\pi_1 + \pi_2 = 1$, the above equation yields an identified system of two equations and two unknowns. The resulting *pseudo* probabilities are then:

$$\begin{cases} \pi_1 = p^{\xi} = \frac{1}{2} - \frac{q\phi x}{2[2]_q!} \frac{D_q^2[u(x)]}{D_q[u(x)]} \\ \\ \pi_2 = (\Omega - p)^{\xi} = \frac{1}{2} + \frac{q\phi x}{2[2]_q!} \frac{D_q^2[u(x)]}{D_q[u(x)]} = 1 - \pi_1 \end{cases} \quad (\xi < 1) \quad (28)$$

The interpretation for the subjective *pseudo* probability in equation (28) remains intact with one major difference, it does not assume completeness as long as $\Omega \geq 1$. We use the fact that $\pi_1 = p^{\xi} = \frac{p}{\Omega}$, where Ω is seen as a partition or normalising function, to deduce from equation (28) that:

$$\Omega(p, \xi) = p^{1-\xi} = \left(\frac{1}{2} - \frac{q\phi x}{2[2]_q!} \frac{D_q^{(2)} [u(x)]}{D_q [u(x)]} \right)^{\frac{1-\xi}{\xi}} \quad (29)$$

The above indicates that the partition function, $\Omega(p, \xi)$, is inversely related to (ξ) . Therefore, this function may be used to infer an agents belief regarding the overall incompleteness of his probabilities (or perceived outcomes).

To demonstrate how background risk affects (given risk aversion, $D_1^{(2)}[u] < 0$) the function in equation (29), we recall the results in the preceding section. We observe, in general, that the subjective probability increases as background risk becomes more significant (q decreases). Equation (29) must, therefore, be increasing as background risk becomes more significant (q decreases). This is also true for the q -absolute risk aversion $\left(-\frac{D_q^{(2)}[u]}{D_q[u]} \right)$, which positively affect $\Omega(p, \xi)$. However, as indicated in figure 2, the q-absolute risk aversion increases as q decreases. We deduce that background risk is possibly an important cause for deformed subjective distributions.

3 Moment Preferences

Brockett and Kahane (1992) determine that the three first statistical moments of aggregate wealth returns are not sufficient to describe an agent preferences. This makes sense considering the possibility of background risk. Determining statistical preferences to characterise agents is important, as these moments are used to build an empirical framework for estimating pricing kernel (CAPM, ICAPM, Merton (1973) CCAPM and other more advanced models)[4]. As in the previous section, the *q-Taylor* expansion is considered up to the fourth moment of aggregate wealth returns.

[4]Campbell et al. (1997) provide a wide discussion on parametric and non-parametric methods to estimate the pricing kernel.

So far, we argued that the use of infinitesimal calculus in the analysis of agents preferences is applicable when no background risk is present. In this case:

$$u\left(\bar{x}_T\right) = u\left(\bar{x}_T\right) + \left(x_T - \bar{x}_T\right) u'\left(\bar{x}_T\right) + \frac{1}{2}\left(x_T - \bar{x}_T\right)^2 u''\left(\bar{x}_T\right)$$
$$\Rightarrow Eu\left(\bar{x}_T\right) = u\left(\bar{x}_T\right) + \frac{1}{2}\sigma_x^2 u''\left(\bar{x}_T\right) \tag{30}$$

Where: x and $\bar{x}$ are aggregate and average aggregate wealth at time T and $u(\cdot)$ is some utility function. For a power utility, equation (30) is reduced to:

$$Eu(\bar{x}_T) = \frac{\bar{x}_T^{1-\gamma}}{1-\gamma} - \frac{1}{2}\gamma(\bar{x}_T^{1-\gamma})\sigma_x^2 \tag{31}$$

Equation (31) indicates that an agent accounts for risk in determining the acceptability of some lotteries or bets. This claim is refuted by Brockett and Kahane (1992) on the premise that for two bets with similar expected value a riskier bet may be preferable in terms of expected utility. This is because the probabilities that are associated with the riskier bet are different in structure. The q-calculus approach relates to this observation in the sense that given the same conditions but different significance level for background risk, the probability of the two bets are different in structure (regardless of their respective risks).

Performing a *q-Taylor* expansion around $\bar{x}$ up to the fourth order (which includes also skewness and kurtosis preferences) and taking its expectations[5] yields:

$$Eu\left(x\right) = u\left(\bar{x}\right) + \sigma_x^2 \frac{D_q^2\left[u(\bar{x})\right]}{\left[2\right]_q!} + S_x \frac{D_q^3\left[u(\bar{x})\right]}{\left[3\right]_q!} + K_x \frac{D_q^4\left[u(x)\right]}{\left[4\right]_q!}$$
$$D_q^2\left[u(\bar{x})\right] < 0 \quad D_q^4\left[u(\bar{x})\right] < 0 \tag{32}$$

Where, S_x and K_x are the respective skewness and kurtosis. The above implies that the more significant background risk is, the more over weight will higher statistical moments be. Hence, given a certain level of risk aversion, background risk causes agents to attach more weights to higher orders of statistical moments. In example 3, a negative exponential utility is cosidered.

Example 3 *Moment Preferences for the negative exponential function*

The n'th order h-derivative of $u(x) = 1 - e^{-\gamma x}$ and their q-derivative counterparts are:

$$D_q^i\left[1 - e^{-\gamma x}\right] = (-1)^{i+1}\left(\frac{1 - q^{-\gamma}}{\ln q}\right) e^{-\gamma x} \tag{33}$$

[5] We note that: $E\left[(x - \bar{x})^i\right] = E\left[(x - \bar{x})_q^i\right]$

Note that the four q-derivatives above diverge when $q \to 0$ $(h \to \infty)$. Using the equation above, equation (30) becomes:

$$Eu(\bar{x}) = 1 - e^{-\gamma\bar{x}} \left\{ 1 + \frac{\sigma_x^2}{[2]_q!}\left(\frac{1-q^{-\gamma}}{\ln q}\right)^2 + \frac{S_x}{[3]_q!}\left(\frac{1-q^{-\gamma}}{\ln q}\right)^3 + \frac{K_x}{[4]_q!}\left(\frac{1-q^{-\gamma}}{\ln q}\right)^4 \right\} \quad (34)$$

For demonstration sake, we assume that $\bar{x} = 5$, $\sigma_x^2 = 1$, $S_x = 0$, and $K_x = 25$. Furthermore, let the coefficient of risk aversion $(\gamma) = 1$. Figure 6 plots equation (34) as a function of q of these values.

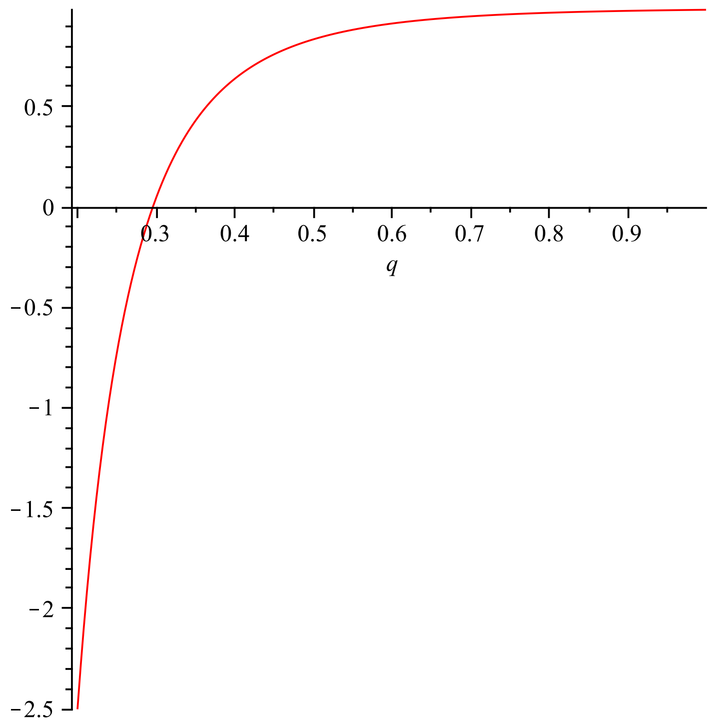

Figure 6: $Eu(x)$-Equation (34)-as a function of q

The expected utility in equation (34) increases with q, i.e., background risk is less significant. As equation (32) indicates, the weights attached to the variance and kurtosis are decreasing when q is increasing. Note that the curve in figure 6 shifts upward as the risk aversion parameter (γ) decreases.

Example 3 illustrates that that given a level of risk-aversion, background risk causes disutility. This is indicated in the simple mathematical relationship (previously stated) where both the variance and the kurtosis are overweighed relative to the skewness parameter. Hence, q can be regarded as a parameter measuring the relative importance of the latter statistical measures. The q

parameter (depicting significance of background risk) can also be interpreted as agents valuation measure of statistical moments.

4 Conclusion

Integrating background risk in a q-calculus analysis framework is the principal contribution of this work. It introduces a mathematical treatment of information incompleteness and utility imprecision, which are well known problems in finance. In particular, q-calculus provides a theoretical framework relating a number of problems where background risk is prevalent. For example, relating liquidity to (latent) background risk. Which enable to assess its price in terms of its background risk. Naturally, other problems can be considered as well. Hence it provides a fertile ground for future research.

The q-calculus approach to risk aversion, demonstrates that statistical moments are not sufficient to describe agents preferences under incomplete state preferences (or imprecision of utility assignment). These states may then be either over or under weighted relative to their true value. In some instances the risk measured might be misleading when adding a possible background risk. Further, background risk is proved to increase risk-aversion, making it more sensitive to changes in wealth.

The appeal of q-calculus, has some drawbacks however. First and most obvious, is its dealing with more complex utility functions. For example: the Epstein and Zin (1989) utility function. Hence, it is difficult to obtain meaningful results without assuming a specific functional and parametric utility function. On the other hand, limiting the analysis to simple functional forms may also be misleading. For example: applying q-calculus to a simple power utility function, where an asset-pricing kernel is derived, is perhaps misleading in presuming that the effects of background risk may disappear in CCAPM pricing kernel. Nevertheless, it can be circumvented by means of the q-Taylor expansion on the asset-pricing kernel.

A number of research venues are opened using the q-calculus approach. A first possibility consists in verifying wether and how does the parameter q relates to “non-extensive” systems that are characterised by long-term memory and long-range correlations that are not accounted for in standard models. Second, it may be interesting to consider how it can be incorporated in behavioural finance. Similarly, the q-parameter may found to be associated to some important concepts in finance such as asset prices, liquidity, price speculations as well as factors contributing or resulting from uncertainty. Finally, q-calculus may be used to better appreciate Knightian uncertainty and its defining parameters in asset-pricing models.

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